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Harmonic functions of $su_q(2)$ for $q \in \mathbb{R}$ and $q \to S^1$

M Novaes, J E M Hornos and E S Bernardes

Departamento de Física e Ciência dos Materiais, Instituto de Física de São Carlos, Universidade de São Paulo, Av. do Trabalhador São-carlense, 400 CP 369 13560.970 São Carlos, SP Brazil

E-mail: sousa@if.sc.usp.br

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Abstract

We show in this paper that a particular family of Askey–Wilson polynomials can be interpreted directly in the light of q-deformed $su_q(2)$ algebras. This approach allows us to correct previous results concerning the q-Legendre functions investigated in Granovskii and Zhedanov (1993 *J. Phys. A: Math. Gen.* **26** 4331). We also establish the orthonormalization and the special cases $q \to 1$ (classical limit) and $q \to \infty$ (asymptote). We conclude that these q-Legendre functions differ significantly from their classical counterparts only when q is in the vicinity of the unitary circle, where the singular points of the absolute value of these q-functions undergo a series of bifurcations.

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1. Introduction

The properties of deformed algebras have been under intense investigation in the last two decades, the main efforts being mainly directed to the general mathematical properties and to applications in integrable systems in statistical mechanics and quantum field theory [1–5]. The possibility of applications of the representation theory of the deformed algebras in modelling physical problems is restricted by a limitative result pointing that the dimensions of the deformed representations are the same as their classical counterparts if the deformation parameter is not a root of unity. In practice, this means that no one expects a substantial diversity of new results from the deformed irreducible representation for the modelling of physical, chemical or biological systems. The main goal of this paper is to investigate the properties of the deformed harmonic functions of the simplest deformed Lie algebra, $su_q(2)$, having in mind possible physical applications and optimization procedures. In order to do this, one needs explicit unitary orthogonality relations and to know the behaviour of the deformed functions realizing the irreducible representations near the unitary circle, including the roots of unity, since only in this case are new possibilities for applications expected.

A few years ago, two different realizations of the quantum algebra $su_q(2)$ on a sphere were introduced simultaneously by Rideau and Winternitz [6] and by Granovskii and Zhedanov [7]. Rideau and Winternitz constructed a basis for its irreducible representations in terms of the so-called q-Vilenkin functions related to the little q-Jacobi [6]. Their work was completed by Irac-Astaud and Quesne [8] where the orthonormality relations of the q-Vilenkin functions were deduced. Recently, yet another realization quite similar to [6] was developed in [9], where the scalar product involves deformed integration.

The realization used by Granovskii and Zhedanov [7] allowed them to construct eigenfunctions to the integer irreducible representations only, which were called spherical q-functions or q-Legendre functions. Employing their Askey–Wilson AW(3) algebraic technique, they also showed that the q-Legendre functions are related to the Askey–Wilson q-deformed polynomials [10] and presented a conjecture for the orthogonality relation (without the normalization constant) satisfied by the q-Legendre functions for $q \in \mathbb{R}^+$, although this last observation is not explicitly stated in [7].

In section 2, the classical Legendre functions and their relation with the classical $su(2)$ algebra is briefly presented. In section 3, the realization proposed in [7] is concisely reviewed. The q-Legendre functions are re-derived in section 4 in order to fix some factors present in [7] and show that these functions are naturally related to the Askey–Wilson q-deformed polynomials, with no need for referring to the AW(3) algebra. The orthonormality relation is discussed in section 5, and we obtain explicitly the normalization constant. In section 6 the special cases $q \to \infty$ (the asymptotic limit) and $q \to 1$ (the classical limit) are considered as well as numerical examples near the unitary circle. We believe these improvements are important not only from the point of view of the q-functions themselves [11–13] but also from that of applications in physics.

2. The classical $su(2)$ algebra

The classical compact Lie algebra $su(2)$ is defined by the following commutation relations:

$$\begin{aligned} &[J_+, J_-] = 2J_z \\ &[J_z, J_\pm] = \pm J_\pm. \end{aligned} \tag{1}$$

Each of its unitary irreducible representations is spanned by the $2j+1$ vectors $|j, m\rangle$ for integer or semi-integers j, $-j \leqslant m \leqslant j$. Using the unitary conditions $J_z^\dagger = J_z$ and $J_\pm^\dagger = J_\mp$, the corresponding matrix elements are given by

$$J^2|j,m\rangle = j(j+1)|j,m\rangle \tag{2}$$

$$J_z|j,m\rangle = m|j,m\rangle \tag{3}$$

$$J_\pm|j,m\rangle = \sqrt{(j\mp m)(j\pm m+1)}|j,m\pm 1\rangle \tag{4}$$

where J^2 is the Casimir operator:

$$J^2 = J_z^2 + \tfrac{1}{2}(J_+J_- + J_-J_+). \tag{5}$$

For integer $j = l$, the $su(2)$ algebra can be realized by the spherical differential operators

$$L_z = -\mathrm{i}\frac{\partial}{\partial\phi} \tag{6}$$

$$L_\pm = \mathrm{e}^{\pm\mathrm{i}\phi}\left(\mathrm{i}\cot\theta\frac{\partial}{\partial\phi} \pm \frac{\partial}{\partial\theta}\right) \tag{7}$$

and the eigenvectors $|l, m\rangle$ become the spherical harmonics [14]

$$Y_{l,m}(\theta, \phi) = \langle \theta, \phi | l, m \rangle = (-1)^m \, \mathrm{e}^{\mathrm{i}m\phi} \left[\frac{(2l+1)}{4\pi} \frac{(l-m)!}{(l+m)!} \right]^{1/2} P_{l,m}(x) \tag{8}$$

where $P_{l,m}(x)$, $x = \cos\theta$, are the associated Legendre functions which are orthogonal with the weight function $\sin\theta$,

$$\int_0^\pi \mathrm{d}\theta \sin\theta \, P_{l,m} P_{l',m} = \delta_{ll'} \frac{2}{2l+1} \frac{(l+m)!}{(l-m)!} \tag{9}$$

and obey the Sturm–Liouville equation given by L^2 and L_z:

$$\frac{1}{\sin\theta} \frac{\mathrm{d}}{\mathrm{d}\theta} \left[\sin\theta \frac{\mathrm{d}}{\mathrm{d}\theta} P_{l,m}(x) \right] + \left[l(l+1) - \frac{m^2}{\sin^2\theta} \right] P_{l,m}(x) = 0. \tag{10}$$

3. The deformed $su_q(2)$ algebra

In the following, we use calligraphic letters for the q-deformed quantities such as the algebra elements, spherical harmonics and Legendre functions. The q-deformed $su_q(2)$ is defined by [3]

$$[\mathcal{J}_z, \mathcal{J}_\pm] = \pm \mathcal{J}_\pm \tag{11}$$

$$[\mathcal{J}_+, \mathcal{J}_-] = [2\mathcal{J}_z]_q \tag{12}$$

where our q-deformation is

$$[x]_q = \frac{q^x - q^{-x}}{q - q^{-1}} = \frac{\sinh \omega x}{\sinh \omega} \qquad q = \mathrm{e}^{\omega}. \tag{13}$$

The classical $su(2)$ algebra is recovered in the limit $q \to 1$, which will be called the classical limit. The irreducible representations for $su_q(2)$ can still be labelled by the same quantum numbers $|l, m\rangle$ but with q-deformed matrix elements [3]:

$$\mathcal{J}^2 |l, m\rangle = [l]_q [l+1]_q |l, m\rangle \tag{14}$$

$$\mathcal{J}_z |l, m\rangle = m |l, m\rangle \tag{15}$$

$$\mathcal{J}_\pm |l, m\rangle = \sqrt{[l \mp m]_q [l \pm m + 1]_q} \, |l, m \pm 1\rangle \tag{16}$$

where

$$\mathcal{J}^2 = [\mathcal{J}_z]_q^2 + \tfrac{1}{2}(\mathcal{J}_+\mathcal{J}_- + \mathcal{J}_-\mathcal{J}_+). \tag{17}$$

When $q \in \mathbb{R}^+$, we can impose $\mathcal{J}_z^\dagger = \mathcal{J}_z$ and $\mathcal{J}_\pm^\dagger = \mathcal{J}_\mp$, as required in physics [3], and the matrix elements in (16) are invariant by $q \to q^{-1}$.

In this work, we are interested only in integer angular momentum $j = l$, in which case we denote the generators $\{\mathcal{L}_z, \mathcal{L}_\pm\}$. The q-spherical differential realization will be done by the following shift operators [7]:

$$\mathcal{L}_z = -\mathrm{i} \frac{\partial}{\partial \phi} \tag{18}$$

$$\mathcal{L}_\pm = \frac{\mathrm{e}^{\pm \mathrm{i}\phi}}{\sinh\omega} \left[\mathrm{i} \cot\theta \cos\left(\omega \frac{\partial}{\partial\theta}\right) \sin\left(\omega \frac{\partial}{\partial\phi}\right) \pm \sin\left(\omega \frac{\partial}{\partial\theta}\right) \cos\left(\omega \frac{\partial}{\partial\phi}\right) \right]. \tag{19}$$

These deformed operators have the classical operators (6) and (7) as their classical limit ($q \to 1$ or $\omega \to 0$). Equations (14) and (15)) can be written in this realization as

$$D_q[A(\theta)D_q\mathcal{Y}_{l,m}(\theta,\phi)] + \left([l]_q[l+1]_q - B(\theta)[m]_q^2\right)\mathcal{Y}_{l,m}(\theta,\phi) = 0 \tag{20}$$

where D_q is the q-deformed derivative [10, 15] in $x = \cos(\theta)$,

$$D_q F(\theta) \equiv \frac{F(\theta+\mathrm{i}\omega) - F(\theta-\mathrm{i}\omega)}{\cos(\theta+\mathrm{i}\omega) - \cos(\theta-\mathrm{i}\omega)} = \frac{F(\theta+\mathrm{i}\omega) - F(\theta-\mathrm{i}\omega)}{-2\mathrm{i}\sinh\omega\sin\theta} \tag{21}$$

and

$$A(\theta) = \sin^2\theta + \sinh^2 m\omega \tag{22}$$

$$B(\theta) = \frac{\cosh\omega}{\sin^2\theta + \sinh^2\omega} + 1 - \cosh\omega. \tag{23}$$

Note that the classical limit of the q-deformed derivative D_q is

$$\lim_{q\to 0} D_q = -\frac{1}{\sin\theta}\frac{\mathrm{d}}{\mathrm{d}\theta} = \frac{\mathrm{d}}{\mathrm{d}x} \qquad x = \cos\theta \tag{24}$$

and therefore equation (10) is the classical limit of equation (20).

4. The *q*-deformed spherical harmonics

Following Granovskii and Zhedanov [7], instead of attempting to solve (20), we investigate the action of $\mathcal{L}_+$ on the eigenfunctions $\mathcal{Y}_{l,m} = \langle\theta,\phi|l,m\rangle$ in order to find an explicit q-deformed function for $\mathcal{Y}_{l,m}$. First, we assume the usual variable separation:

$$\mathcal{Y}_{l,m}(\theta,\phi) = \mathrm{e}^{\mathrm{i}m\phi}\Psi_{l,m}(\theta) \qquad m \geqslant 0. \tag{25}$$

Using the representation given in (19), equation (16) can be written as

$$\frac{\Psi_{l,m}(\theta+\mathrm{i}\omega)\sin(\theta-\mathrm{i}m\omega) - \Psi_{l,m}(\theta-\mathrm{i}\omega)\sin(\theta+\mathrm{i}m\omega)}{2\mathrm{i}\sinh\omega\sin\theta} = \sqrt{[l-m]_q[l+m+1]_q}\,\Psi_{l,m+1}(\theta). \tag{26}$$

We assume also the following functional form for $\Psi_{l,m}(\theta)$:

$$\Psi_{l,m}(\theta) = C_{l,m}\zeta_m(\theta)\Phi_{l,m}(\theta) \tag{27}$$

where $C_{l,m}$ is a constant which we choose, for convenience, as

$$C_{l,m} = \frac{(-1)^m}{[2m]_q!!}\left(\frac{[2l+1]_q}{4\pi}\frac{[l+m]_q!}{[l-m]_q!}\right)^{1/2}\prod_{k=m+1}^{l}\cosh(k\omega) \tag{28}$$

and the functions $\Phi_{l,m}(\theta)$ and $\zeta_m(\theta)$ are to be determined. The q-deformed factorial $[k]_q!$ and the double factorial $[k]_q!!$ are defined as follows:

$$[k]_q! = [k]_q[k-1]_q\cdots \tag{29}$$

$$[k]_q!! = [k]_q[k-2]_q\cdots \tag{30}$$

The $\zeta_m(\theta)$ function can be determined by taking $l = m$ in (26) and imposing $\Phi_{l,m}(\theta)$ to be a polynomial of degree $l - m$ in $\cos\theta$. The resulting equation,

$$\zeta_m(\theta+\mathrm{i}\omega)\sin(\theta-\mathrm{i}m\omega) = \zeta_m(\theta-\mathrm{i}\omega)\sin(\theta+\mathrm{i}m\omega) \tag{31}$$

is solved by

$$\zeta_m(\theta) = g_m(\theta) F_m(\theta) \tag{32}$$

where

$$\begin{aligned} F_m(\theta) &= \prod_{k=0}^{m-1} \sin(\theta + \mathrm{i}\omega(m-1-2k)) \\ &= \begin{cases} \prod_{k=1}^{m/2} (\cosh^2(2k-1)\omega - \cos^2\theta) & \text{for} \quad m \text{ even} \\ 1 & \text{for} \quad m = 0 \\ \sin\theta \prod_{k=1}^{(m-1)/2} (\cosh^2 2k\omega - \cos^2\theta) & \text{for} \quad m \text{ odd} \end{cases} \end{aligned} \tag{33}$$

and $g_m(\theta)$ is any function of θ with period $2\mathrm{i}\omega$ besides the natural real period 2π. As we will see, this double-periodic function is related to the Jacobi theta functions and plays the role of a q-deformed weight function. Note that

$$F_m^2(\theta) = 2^{-2m} (q^{-2}\, \mathrm{e}^{2\mathrm{i}(\theta + \mathrm{i}m\omega)}, q^{-2}\, \mathrm{e}^{-2\mathrm{i}(\theta+\mathrm{i}m\omega)}; q^{-4})_m \tag{34}$$

where $(a;q)_k$ is the q-deformed Pochhammer symbol

$$(a;q)_k = \begin{cases} (1-a)(1-aq)\cdots(1-aq^{k-1}) & \text{if} \quad k = 1, 2, \ldots, \\ 1 & \text{if} \quad k = 0 \end{cases} \tag{35}$$

and

$$(a, b; q)_k = (a;q)_k (b;q)_k. \tag{36}$$

Setting the m dependence in $g_m(\theta)$ as

$$g_m(\theta) = g(\theta + \mathrm{i}m\omega). \tag{37}$$

Equation (26) becomes

$$D_q \Phi_{l,m}(\theta) = \frac{[l+m+1]_q [l-m]_q}{2[m+1]_q \cosh^2(m+1)\omega} \Phi_{l,m+1}(\theta). \tag{38}$$

The solution to this equation is the following basic q-hypergeometric polynomial (A complete proof is presented in appendix A.):

$$\Phi_{l,m}(\theta|q) = {}_4\varphi_3 \left(\begin{matrix} a_1, a_2, a_3, a_4 \\ b_1, b_2, b_3 \end{matrix} ; q^{-2}, q^{-2} \right) = \sum_{k=0}^{l-m} \frac{\prod_{s=1}^4 (a_s; q^{-2})_k}{\prod_{r=1}^3 (b_r; q^{-2})_k} \frac{q^{-2k}}{(q^{-2}; q^{-2})_k} \tag{39}$$

with

$$\begin{aligned} &a_1 = q^{2(l-m)} && a_2 = q^{-2(l+m+1)} && a_3 = \mathrm{e}^{\mathrm{i}\theta} q^{-(m+1)} \\ &a_4 = \mathrm{e}^{-\mathrm{i}\theta} q^{-(m+1)} && b_1 = -b_2 = -b_3 = q^{-2(m+1)}. \end{aligned} \tag{40}$$

Note that $\Phi_{l,m}$ is invariant under $q \to q^{-1}$ and has only even or odd powers of $\cos\theta$ [10]. These polynomials are related to the Askey–Wilson [10] q-deformed polynomials $p_n(x|q)$:

$$p_n(x; a, b, c, d|q^{-2}) = \Xi_{l,m} \Phi_{l,m}(x|q) \tag{41}$$

$$a = b = -c = -d = q^{-(m+1)} \qquad n = l - m \quad x = \cos\theta \tag{42}$$

$$\Xi_{l,m} = q^{(l-m)(m+1)} (q^{-2(m+1)}; q^{-2})_{l-m} (-q^{-2(m+1)}; q^{-2})_{l-m}^2. \tag{43}$$

For $m = 0$, the $\Phi_{l,m}(x|q)$ polynomials, up to a constant factor, are the special case $\alpha = \beta = 0$ of the q-Jacobi polynomials [10, equation (4.17)] defined by Rahman [16] with $q \to q^{-2}$. See also [13, 17–20, 12] for another interpretation of these polynomials as q-deformed spherical functions. Since the Askey–Wilson polynomials $p_n(x|q)$ satisfy a general three term recurrence relation,

$$2xp_n(x) = A_n p_{n+1}(x) + B_n p_n(x) + C_n p_{n-1}(x) \qquad p_{-1} = 0 \quad p_0 = 1 \tag{44}$$

there is a corresponding relation for $\Phi_{l,m}$ involving different l and the same m. For the present case, the coefficients A_n, B_n and C_n are

$$\begin{aligned} A_n &= \frac{q^{3l-m+2}}{2\sinh\omega}\frac{[l+m+1]_q}{[2l+1]_q[2l+2]_q} \\ B_n &= 0 \\ C_n &= 8q^{-3l+m+1}\cosh^2 l\omega\sinh\omega\frac{[l-m]_q[2l]_q}{[2l+1]_q} \end{aligned} \tag{45}$$

and therefore equation (44) becomes

$$\cos\theta\,\Phi_{l,m} = \cosh(l+1)\omega\frac{[l+m+1]_q}{[2l+1]_q}\Phi_{l+1,m} + \cosh l\omega\frac{[l-m]_q}{[2l+1]_q}\Phi_{l-1,m}. \tag{46}$$

Gathering all information obtained so far, the q-spherical harmonics (25) can be written as

$$\mathcal{Y}_{l,m}(\theta,\phi) = (-1)^m\,\mathrm{e}^{\mathrm{i}m\phi}\left(\frac{[2l+1]_q}{4\pi}\frac{[l-m]_q!}{[l+m]_q!}\right)^{1/2}\mathcal{P}_{l,m}(\theta) \tag{47}$$

with

$$\mathcal{P}_{l,m}(\theta) = \Upsilon_{l,m}F_m(\theta)g_m(\theta)\Phi_{l,m}(\cos\theta) \tag{48}$$

and

$$\Upsilon_{l,m} = \frac{1}{[2m]_q!!}\frac{[l+m]_q!}{[l-m]_q!}\prod_{k=m+1}^{l}\cosh k\omega. \tag{49}$$

Therefore, replacing $\Phi_{l,m}$ by $\mathcal{P}_{l,m}$ in (46) and setting $x = \cos\theta$, we have

$$[2l+1]_q x\mathcal{P}_{l,m}(x) = [l-m+1]_q\mathcal{P}_{l+1,m}(x) + \cosh^2 l\omega[l+m]_q\mathcal{P}_{l-1,m}(x). \tag{50}$$

It is shown in section 6 that the classical associated Legendre functions $P_{l,m}$ are the classical limit ($q \to 1$) of $\mathcal{P}_{l,m}$, which will be called q-deformed associated Legendre functions.

5. The orthogonality relation

An orthogonality relation for the q-deformed associated Legendre functions can be obtained directly from the properties of the Askey–Wilson polynomials. Two cases must be considered: $q \in \mathbb{R}$ and $q \in \mathbb{C}$. In the real case, unitary representations can be defined as usual [3]. For the complex case, there are limitations to unitarize the irreducible representations even in the unitary circle [8] as will be seen.

For Re $q > 1$ we have [10, p 14, theorem 2.5]

$$\frac{1}{2\pi}\int_0^\pi \mathrm{d}\theta\, w(\theta)p_{l-m}(\theta|q^{-2})p_{l'-m}(\theta|q^{-2}) = \delta_{l,l'}h_{l,m} \tag{51}$$

since there are no point masses such that $|q^{-(m+1)}q^{-2k}| > 1$, $k \geqslant 0$, once relation (41) is taken into account. The weight function $w(\theta)$ for the present case is given by

$$\begin{aligned} w(\theta) &= \frac{(\mathrm{e}^{2\mathrm{i}\theta}, \mathrm{e}^{-2\mathrm{i}\theta}; q^{-2})_\infty}{(q^{-(m+1)}\mathrm{e}^{\mathrm{i}\theta}, q^{-(m+1)}\mathrm{e}^{-\mathrm{i}\theta}; q^{-2})^2_\infty(-q^{-(m+1)}\mathrm{e}^{\mathrm{i}\theta}, -q^{-(m+1)}\mathrm{e}^{-\mathrm{i}\theta}; q^{-2})^2_\infty} \\ &= \frac{2^{2m+1}}{q^{m^2-\frac{1}{2}}}\sin\theta F_m^2(\theta)\frac{\vartheta_1(\theta+\mathrm{i}m\omega|q^{-2})}{\vartheta_4(\theta+\mathrm{i}m\omega|q^{-2})} \end{aligned} \tag{52}$$

where we have used (34) and ϑ_k are the Jacobi theta functions [21]:

$$\vartheta_1(z|q) = 2q^{1/4} \sin z (q^2, q^2 \,\mathrm{e}^{2\mathrm{i}z}, q^2 \,\mathrm{e}^{-2\mathrm{i}z}; q^2)_\infty \tag{53}$$

$$\vartheta_2(z|q) = 2q^{1/4} \cos z (q^2, -q^2 \,\mathrm{e}^{2\mathrm{i}z}, -q^2 \,\mathrm{e}^{-2\mathrm{i}z}; q^2)_\infty \tag{54}$$

$$\vartheta_3(z|q) = (q^2, -q \,\mathrm{e}^{2\mathrm{i}z}, -q \,\mathrm{e}^{-2\mathrm{i}z}; q^2)_\infty \tag{55}$$

$$\vartheta_4(z|q) = (q^2, q \,\mathrm{e}^{2\mathrm{i}z}, q \,\mathrm{e}^{-2\mathrm{i}z}; q^2)_\infty. \tag{56}$$

The normalization constant in (51) is given by

$$\begin{aligned} h_{l,m} &= \frac{(q^{-4(l+1)}; q^{-2})_\infty (q^{-2(l+m+1)}; q^{-2})_{l-m}}{(-q^{-2(l+1)}; q^{-2})_\infty^4 (q^{-2(l+1)}; q^{-2})_\infty^2 (q^{-2(l-m+1)}; q^{-2})_\infty} \\ &= \frac{2q^{2l+\frac{1}{2}}}{\pi} \frac{(q^{-4}; q^{-4})_l^2 (-q^{-2}; q^{-2})_l^2}{(q^{-2(l-m+1)}; q^{-2})_{2m}} M(\omega) \end{aligned} \tag{57}$$

where

$$M(\omega) = \frac{\pi}{\sinh \omega \vartheta_2^2(0|\,\mathrm{e}^{-\omega})}. \tag{58}$$

We impose the usual relation

$$\int_0^\pi \mathrm{d}\theta \sin\theta \mathcal{P}_{l,m} \mathcal{P}_{l',m} = \delta_{l,l'} \mathcal{N}_{l,m} \tag{59}$$

(note that there is no complex conjugation) with

$$\mathcal{N}_{l,m} = 2\pi h_{l,m} \frac{q^{m^2-\frac{1}{2}}}{2^{2m+1}} \left(\frac{\Upsilon_{l,m}}{\Xi_{l,m}} \right)^2 = \frac{2}{[2l+1]_q} \frac{[l+m]_q!}{[l-m]_q!} M(\omega) \prod_{r=1}^{l} \cosh^2 r\omega \tag{60}$$

in order to obtain the g_m function. The final result is

$$g_m^2(\theta|\omega) = \frac{\vartheta_1(\theta + \mathrm{i}m\omega|\mathrm{e}^{-2\omega})}{\vartheta_4(\theta + \mathrm{i}m\omega|\mathrm{e}^{-2\omega})} = \begin{cases} \sqrt{\kappa}\,\mathrm{sn}\,(2K\theta/\pi|\kappa) & \text{for} \quad m \text{ even} \\ [\sqrt{\kappa}\,\mathrm{sn}(2K\theta/\pi|\kappa)]^{-1} & \text{for} \quad m \text{ odd} \end{cases} \tag{61}$$

where sn is the elliptic sine, and κ is the elliptic modulus and $K = K(\kappa)$ is the complete elliptic integral of the first kind [21]:

$$\kappa = \frac{\vartheta_2^2(0|\mathrm{e}^{-2\omega})}{\vartheta_3^2(0|\mathrm{e}^{-2\omega})} \qquad K(\kappa) = \frac{\pi}{2} \vartheta_3^2(0|\mathrm{e}^{-2\omega}). \tag{62}$$

It is important to note that the scalar product (59) works for $q \in \mathbb{C}$ as well as for $q \in \mathbb{R}$, but only in the latter case can the irreducible representations of $su_q(2)$ be made unitary [3]. Note also that no exotic q-deformed integration is present in (59).

As a final remark, note that the presence of the elliptic sine in (61) brings about many unique features to these q-functions. For example, for real deformations when $\omega \to 0$, $g_m(\theta|\omega)$ converges weakly to unity, but the endpoints $g_m(0|\omega)$ and $g_m(\pi|\omega)$ are always zero for m even (see figure 1). This property makes the present q-Legendre functions quite different from the q-Vilenkin [8] and the classical Legendre functions in the vicinity of $\theta = 0$ and $\theta = \pi$, as shown in figure 6 (see section 6.3 for further discussions).

6. Special cases

6.1. $q \to 1$

The special case $q \to 1$, or equivalently $\omega \to 0$, corresponds to the classical limit, since the commutation relations (12) reduce to their classical counterparts given in (1). In this section,

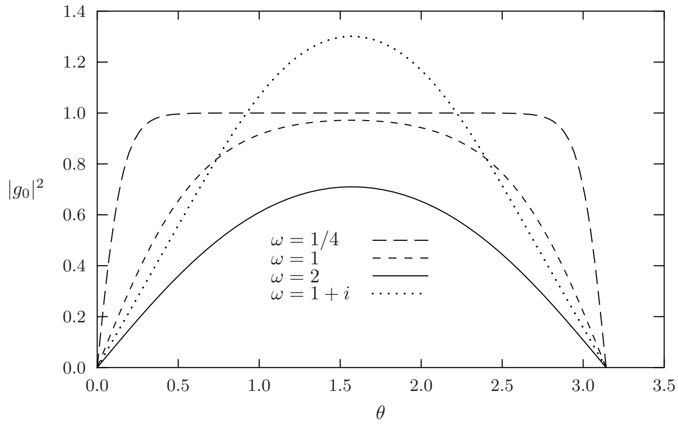


Figure 1. Absolute value of the weight function $g_m(\theta|\omega)$ given in (61), m even, $q = \exp(\omega)$, far away from the unitary circle ($|q| = 1$). For real ω, $g_m^2 = 1$ is the upper bound when $\omega \to 0$.

it is shown that the q-deformed Legendre functions (48) reduce to the classical Legendre functions when $q \to 1$.

First, we calculate the classical limit of the q-hypergeometric polynomial $\Phi_{l,m}$. The deformed Pochhammer symbols appearing in (39) can also be written as

$$(\pm q^{-2\alpha}; q^{-2})_k = q^{-k\alpha} q^{-k(k-1)/2} \prod_{r=0}^{k-1} (q^{\alpha+r} \mp q^{-(\alpha+r)}) \tag{63}$$

where $\alpha \in \{-(l-m), l+m+1, m+1, 1\}$ and

$$(q^{-a}\,\mathrm{e}^{\mathrm{i}\theta}, q^{-a}\,\mathrm{e}^{-\mathrm{i}\theta}; q^{-2})_k = 2^k q^{-ak} q^{-k(k-1)} \prod_{r=0}^{k-1} \{\cosh(a+2r)\omega - \cos\theta\} \tag{64}$$

which are more suitable for taking the limit. Substituting these last expressions back in (39) and using the deformation given in (13), we have

$$\Phi_{l,m}(\theta) = \sum_{k=0}^{l-m-1} (-2)^{-k} \prod_{r=0}^{k-1} \frac{[l-m-r]_q [l+m+1+r]_q}{[m+1+r]_q [r+1]_q} \frac{(\cosh(m+1+2r)\omega - \cos\theta)}{\cosh^2(m+1+r)\omega}. \tag{65}$$

Now we can see immediately that the classical limit for $\Phi_{l,m}$ is a hypergeometric series:

$$\begin{aligned} \lim_{q\to 1} \Phi_{l,m}(\theta) &= \sum_{k=0}^{l-m-1} (-1)^k \frac{(1-\cos\theta)^k}{2^k} \prod_{r=0}^{k-1} \frac{(l-m-r)(l+m+1+r)}{(m+1+r)(r+1)} \\ &= {}_2F_1\left(\begin{matrix} -(l-m), l+m+1 \\ m+1 \end{matrix}; \frac{1-\cos\theta}{2}\right). \end{aligned} \tag{66}$$

Second, we calculate the classical limit of g_m defined in (61). Since $\kappa \to 1$ and $K(\kappa) \to \infty$ when $\omega \to 0$,

$$\lim_{\omega\to 0} \sqrt{\kappa}\, \mathrm{sn}(2K\theta/\pi|\kappa) = 1. \tag{67}$$

It is now easy to see that

$$\lim_{q\to 1} \mathcal{P}_{l,m}(\theta) = \frac{1}{2^m m!} \frac{(l+m)!}{(l-m)!} \sin^m\theta\, {}_2F_1\left(\begin{matrix} -(l-m), l+m+1; \\ m+1 \end{matrix} \frac{1-\cos\theta}{2}\right) = P_{l,m}(\theta) \tag{68}$$

and that the classical limit of (50) is the well-known classical recurrence relation [14] for $P_{l,m}$:

$$(2l+1)xP_{l,m}(x) = (l-m+1)P_{l+1,m}(x) + (l+m)P_{l-1,m}(x). \tag{69}$$

Since $M(\omega) \to 1$ when $\omega \to 0$, the normalization constant (60) also has the appropriate classical limit.

6.2. $q \to \infty$

The asymptotic expansion $q \to \infty$ is another important special case. The following basic results follow directly from the definitions of the respective quantities:

$$[x]_q \to q^{x-1} \qquad [x]_q! \to q^{x(x-1)/2} \qquad [x]_q!! \to q^{x^2} \qquad x > 0 \tag{70}$$

$$\cosh x\omega \to \tfrac{1}{2}q^x \qquad \sinh x\omega \to \tfrac{1}{2}q^x \qquad x \geqslant 0 \quad (\pm q^\alpha; q^\beta)_n \to 1 \quad \alpha < 0 \quad \beta < 0. \tag{71}$$

Thus, it is easy to see that

$$F_m(\theta) \to \begin{cases} 2^{-m}q^{m^2/2} & \text{for} \quad m \text{ even} \\ 2^{1-m}q^{(m^2-1)/2}\sin\theta & \text{for} \quad m \text{ odd} \end{cases} \tag{72}$$

and

$$g_m(\theta) \to \begin{cases} (\sqrt{q})^{-1}2\sin\theta & \text{for} \quad m \text{ even} \\ \sqrt{q}(2\sin\theta)^{-1} & \text{for} \quad m \text{ odd.} \end{cases} \tag{73}$$

Therefore, the weight function (52) and the normalization constant (59) have very simple asymptotes:

$$w(\theta) \to 4\sin^2\theta \qquad \mathcal{N}_{l,m} \to \pi 2^{-2l} q^{l(l-1)+m(2l-1)-1/2}. \tag{74}$$

Now, a normalized asymptotic expansion to the q-deformed Legendre functions (48) can be written as

$$\sqrt{\mathcal{N}_{l,m}}\,\mathcal{P}_{l,m} \overset{q\to\infty}{\to} \sqrt{\frac{2\sin\theta}{\pi}}\,\tilde{p}_{l-m}(\theta) \tag{75}$$

where $\tilde{p}_{l-m}(\theta)$ is the asymptotic expansion of the Askey–Wilson polynomials [10, p 27], given by

$$\tilde{p}_n(\theta) = U_n(\theta) - 2q^{-2(m+1)}U_{n-2}(\theta) + q^{-4(m+1)}U_{n-4}(\theta) \xrightarrow{q\to\infty} U_n(\theta) \tag{76}$$

with

$$\begin{aligned} \tilde{p}_2(\theta) &= U_2(\theta) - 2q^{-2(m+1)} - q^{-4(m+1)} \xrightarrow{q\to\infty} U_2(\theta) \\ \tilde{p}_1(\theta) &= (1 - q^{-4(m+1)})U_1(\theta) \xrightarrow{q\to\infty} U_1(\theta) \\ \tilde{p}_0(\theta) &= 1 \end{aligned} \tag{77}$$

and the Chebyshev polynomials of the second kind

$$U_n(\theta) = \frac{\sin(n+1)\theta}{\sin\theta}. \tag{78}$$

Finally, the normalized asymptotic q-deformed Legendre functions reduce to the Fourier basis with a weight function $\sin\theta$:

$$A_{l,m}(\theta) = \lim_{q\to\infty}\sqrt{\mathcal{N}_{l,m}}\,\mathcal{P}_{l,m} = \sqrt{\frac{2}{\pi}}\frac{\sin(l-m+1)\theta}{\sqrt{\sin\theta}}. \tag{79}$$

6.3. *Near the unitary circle*

For complex values of the deformation parameter

$$q = \exp(\omega) \qquad \omega = \omega_R + \mathrm{i}\omega_I \qquad \omega_R, \omega_I \in \mathbb{R} \tag{80}$$

the function $g_m(\theta|\omega)$ given in equation (61) can present very rich structures, especially as q approaches the unitary circle S^1 ($\omega_R = 0$), in contrast with the case of real q, when no significant deviations from the classical case appear. As a consequence, the q-Legendre functions can lose all similarity with their classical analogues as $q \to S^1$. In the following, we present several numerical examples of this phenomenon. As we will see, the qualitative topographic properties of the q-functions are changed with the control parameter ω_R in a typical morsification process with non-Morse critical points of type A_2 [22].

The q-deformed Legendre and Vilenkin functions are implemented in symbolic computer codes as well as the basic tools for deformation and the numerical examples presented in this section [23].

6.3.1. The weight function g_m. The absolute value $|g_m(\theta|\omega)|^2$, m even, is shown in figure 1 for three different real values of the deformation parameter ω, and one complex value far from the unitary circle. As we can see, there is no topographic difference among these four functions: each one has one maximum. It is also interesting to see that the classical limit $g_0 = 1$ is attained in a non-trivial manner near the endpoints as shown by the long-dashed curve ($\omega = 1/4$) in figure 1.

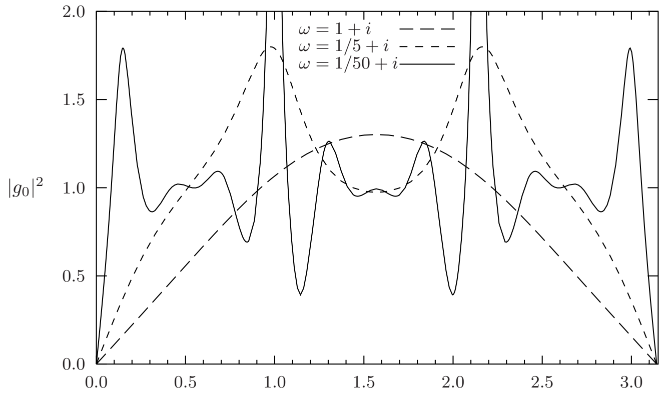
For complex deformations near the unitary circle, $\omega_I \neq 0, \omega_R \to 0$, new maxima and minima of $|g_m(\theta|\omega)|^2$ are unfolded, as shown in figure 2 for m even. This unfolding is controlled by the deformation parameter ω. In figure 2(a), we have a global view ($0 \leqslant \theta \leqslant \pi$) of this behaviour for three values of the deformation parameter. We can see two new maxima when we move from $\omega = 1 + \mathrm{i}$ to $\omega = 1/5 + \mathrm{i}$. Moving a little closer, from $\omega = 1/5 + \mathrm{i}$ to $\omega = 1/50 + \mathrm{i}$, we will find 11 new maxima. As we keep approaching the unitary circle, a proliferation of such critical points takes place, and a typical situation is presented for $\omega = 10^{-5} + \mathrm{i}$ (see figure 2(b), the local view). This phenomenon is typical of an unfolding of singular points (maxima, minima and saddle points) as shown in figures 3 and 4 (the corresponding intensities). Note that this unfolding process has a fractal-like nature, occurring in ever finer scales as the deformation parameter moves towards the unitary circle.

All positions of the singular points in figure 3 can be inferred from (61). For $m = 0$ (m even), we have

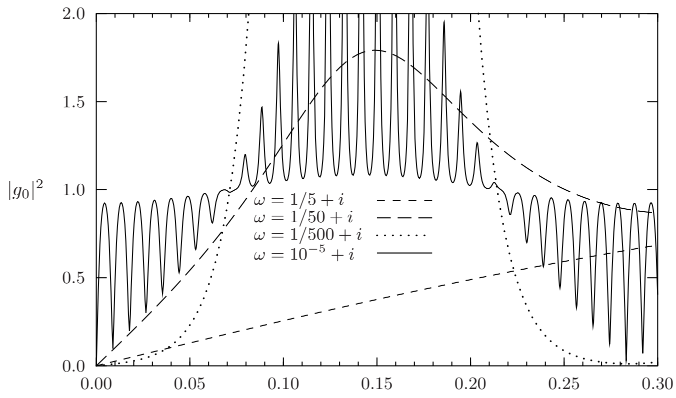
$$g_0^2(\theta|\omega_R + \mathrm{i}\omega_I) = 2\sin\theta\,\mathrm{e}^{-\omega/2} \prod_{k=0}^{\infty} \times \frac{\{1 - \mathrm{e}^{-4\omega_R(k+1)}\,\mathrm{e}^{2\mathrm{i}[\theta - 2\omega_I(k+1)]}\}}{\{1 - \mathrm{e}^{-2\omega_R(2k+1)}\,\mathrm{e}^{2\mathrm{i}[\theta - \omega_I(2k+1)]}\}} \frac{\{1 - \mathrm{e}^{-4\omega_R(k+1)}\,\mathrm{e}^{-2\mathrm{i}[\theta + 2\omega_I(k+1)]}\}}{\{1 - \mathrm{e}^{-2\omega_R(2k+1)}\,\mathrm{e}^{-2\mathrm{i}[\theta + \omega_I(2k+1)]}\}}. \tag{81}$$

From this expression, we can see the maxima (poles) at $\theta = \pm r\pi \pm s\omega_I$, with r and s integers and s odd (the plus and minus signs can be taken independently). The minima (zeros) correspond to even values of s. Some of these maxima are labelled in figure 5, with their (r, s) integers given in table 1. In the limit case $\omega_R \to 0$, corresponding to $q \in S^1$, (81) has an infinite number of zeros and poles. Figures 2 and 3 show this behaviour for m even. Similar analyses are made in the next section for other q-deformed functions.

It is remarkable that the function $|g_m(\theta|0.01 + \mathrm{i})|^2$, for example, can be reproduced quite accurately by a superposition of Gaussians,



(a) Global behaviour ($0 \leq \theta \leq \pi$).



(b) Local behaviour ($0 \leq \theta \leq \pi/8$).

Figure 2. (a) Behaviour of $|g_m(\theta|\omega)|^2$, m even, when the deformation parameter $q = \exp(\omega)$, $\omega = \omega_R + \mathrm{i}\omega_I$, is near the unitary circle ($\omega_R \ll 1$). Its corresponding bifurcation diagram is shown in figure 3. (b) Same as before, but with $\theta < \pi/8$ in order to show the unfolding of the singular points for smaller ω_R in a slow process.

Table 1. Non-optimized parameters of the Gaussian (82). All maxima in figure 5 are labelled by the integers in the second row.

k	1	2	3	4	5	6	7	8	9	10	11
(r, s)	$(1, -3)$	$(3, -9)$	$(-2, 7)$	$(0, 1)$	$(2, -5)$	$(-3, 11)$	$(-1, 5)$	$(1, -1)$	$(3, -7)$	$(-2, 9)$	$(0, 3)$
α_k	150	50	100	300	100	50	100	300	100	50	150

$$\sum_{k=1}^{11} A_k \exp(-\alpha_k(\theta - \theta_k)^2) \qquad \theta_k = r\pi + s \qquad A_k = |g_m(\theta_k|0.01 + \mathrm{i})|^2 \tag{82}$$

as shown in figure 5, with the non-optimized values given in table 1. All maxima shown in figure 5 are labelled by the integers (r, s) given in the second row of table 1. Expression (82), which is typically used in atomic physics as Gaussian orbitals, can be naively interpreted

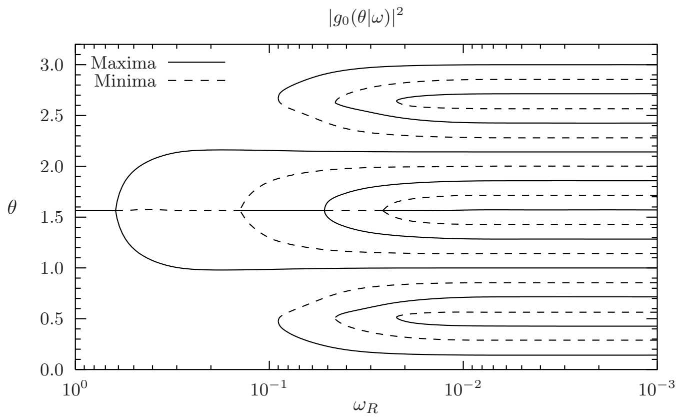


Figure 3. Bifurcation diagram for the singular points of $|g_m(\theta|\omega)|^2$, given in equation (61), for m even, near the unitary circle ($q = \exp(\omega)$, $\omega = \omega_R + \mathrm{i}$). The intensities are shown in figure 4.

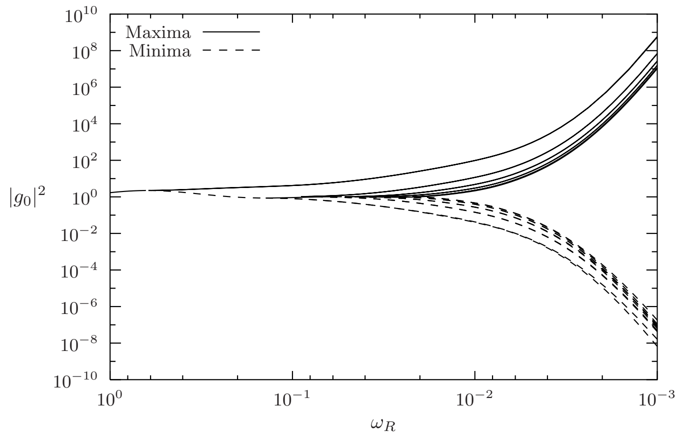


Figure 4. Intensities $|g_0(\theta|\omega_R + \mathrm{i})|^2$ of all maxima and minima shown in figure 3.

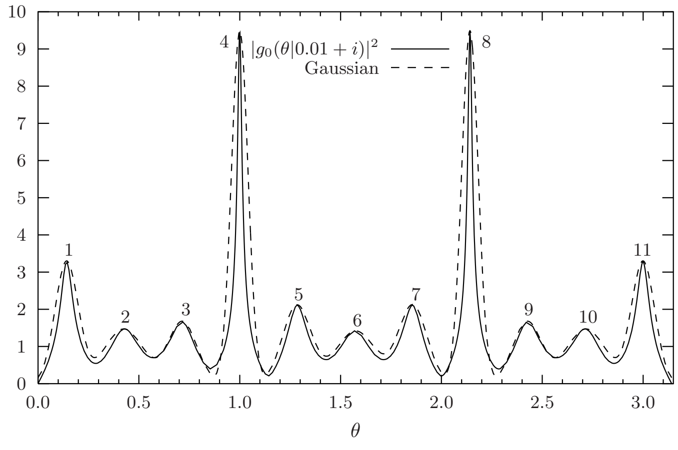


Figure 5. Gaussian approximation of $|g_m(\theta|0.01 + \mathrm{i})|^2$ near the unitary circle. The Gaussian function (dashed lines) is given in (82). All 11 maxima are given in table 1.

Table 2. Un-normalized q-functions $\mathcal{P}_{lm}(x|q)$ (see equations (48) and (61)) up to $l = 3$ where $x = \cos\theta$ and $g_m^{1/2} = \sqrt{g_m(\theta|q)}$. Their normalization constants $\mathcal{N}_{lm}$ and asymptotes $A_{lm}(x|q)$ are shown in table 3.

l	m	$\mathcal{P}_{lm}$
0	0	$g_0^{1/2}$
1	0	$g_0^{1/2}x$
1	1	$g_1^{1/2}\sqrt{1-x^2}$
2	0	$\frac{1}{4}g_0^{1/2}[4(q^4+q^2+1)x^2-(q^2+1)^2]q^{-1}(q^2+1)^{-1}$
2	1	$g_1^{1/2}x\sqrt{1-x^2}q^{-2}(q^4+q^2+1)$
2	2	$-\frac{1}{4}g_2^{1/2}[4q^2x^2-(q^2+1)^2]q^{-4}(q^4+q^2+1)$
3	0	$\frac{1}{4}g_0^{1/2}x[4(q^8+q^6+q^4+q^2+1)x^2-(2q^8+3q^6+2q^4+3q^2+2)]q^{-3}(q^2+1)^{-1}$
3	1	$\frac{1}{4}g_1^{1/2}\sqrt{1-x^2}[4(q^8+q^6+q^4+q^2+1)x^2-(q^4+1)^2]q^{-5}(q^2+1)^{-1}(q^4+q^2+1)$
3	2	$-\frac{1}{4}g_2^{1/2}x[4q^2x^2-(q^2+1)^2]q^{-8}(q^4+q^2+1)(q^8+q^6+q^4+q^2+1)$
3	3	$-\frac{1}{4}g_3^{1/2}\sqrt{1-x^2}[4q^2x^2-(q^4+1)^2]q^{-10}(q^4+q^2+1)(q^8+q^6+q^4+q^2+1)$

Table 3. The normalization constants $\mathcal{N}_{lm}$, equation (60), and asymptotes $A_{lm}(x|q)$, equation (79), up to $l = 3$ for the q-functions $\mathcal{P}_{lm}(x|q)$ shown in table 2, where $C_l = M(q)\prod_{r=1}^{l}(q^{2r}+1)^2$ $(\sum_{s=0}^{2l} q^{2s})^{-1}$, $M(q)$ is the function given in (58) and $B(x) = \sqrt{\frac{2}{\pi}}(1-x^2)^{1/4}$, $x = \cos\theta$.

l	m	$\mathcal{N}_{lm}$	A_{lm}
0	0	$2C_0$	1
1	0	$2C_1$	$2x$
1	1	$2C_1q^{-1}(q^2+1)$	1
2	0	$8C_2q^{-2}$	$(4x^2-1)$
2	1	$8C_2q^{-5}\prod_{j=1}^{2}\left(\sum_{k=0}^{j}q^{2k}\right)$	$2x$
2	2	$8C_2q^{-8}\prod_{j=1}^{3}\left(\sum_{k=0}^{j}q^{2k}\right)$	1
3	0	$32C_3q^{-6}$	$4x(2x^2-1)$
3	1	$32C_3q^{-11}\prod_{j=2}^{3}\left(\sum_{k=0}^{j}q^{2k}\right)$	$(4x^2-1)$
3	2	$32C_3q^{-16}\prod_{j=1}^{4}\left(\sum_{k=0}^{j}q^{2k}\right)$	$2x$
3	3	$32C_3q^{-21}\prod_{j=1}^{5}\left(\sum_{k=0}^{j}q^{2k}\right)$	1

as an approximation to the probability density of a particle trapped in a chain of harmonic potentials.

6.3.2. The q-Legendre functions. In tables 2 and 3 we present a complete list with all q-Legendre functions $\mathcal{P}_{lm}$ (see equations (48) and (61), up to $l = 3$. The q-Vilenkin functions [8] $\mathcal{V}_{lm}$ for $l = 3$ are given in table 4. Both $\mathcal{P}_{lm}$ and $\mathcal{V}_{lm}$ functions are shown in figure 6, for $l = 3$ and $q \in \mathbb{R}$. There are two main differences between these two classes of orthogonal q-functions for real q. First, unlike the q-Vilenkin functions, the q-Legendre functions have a definite parity in $\theta = \pi/2$. Second, their behaviour as $q \to \infty$ are completely different: the $\mathcal{P}_{lm}$ have a well defined limit (as discussed in section 6.2), while the $\mathcal{V}_{lm}$ diverge. It must be remarked that

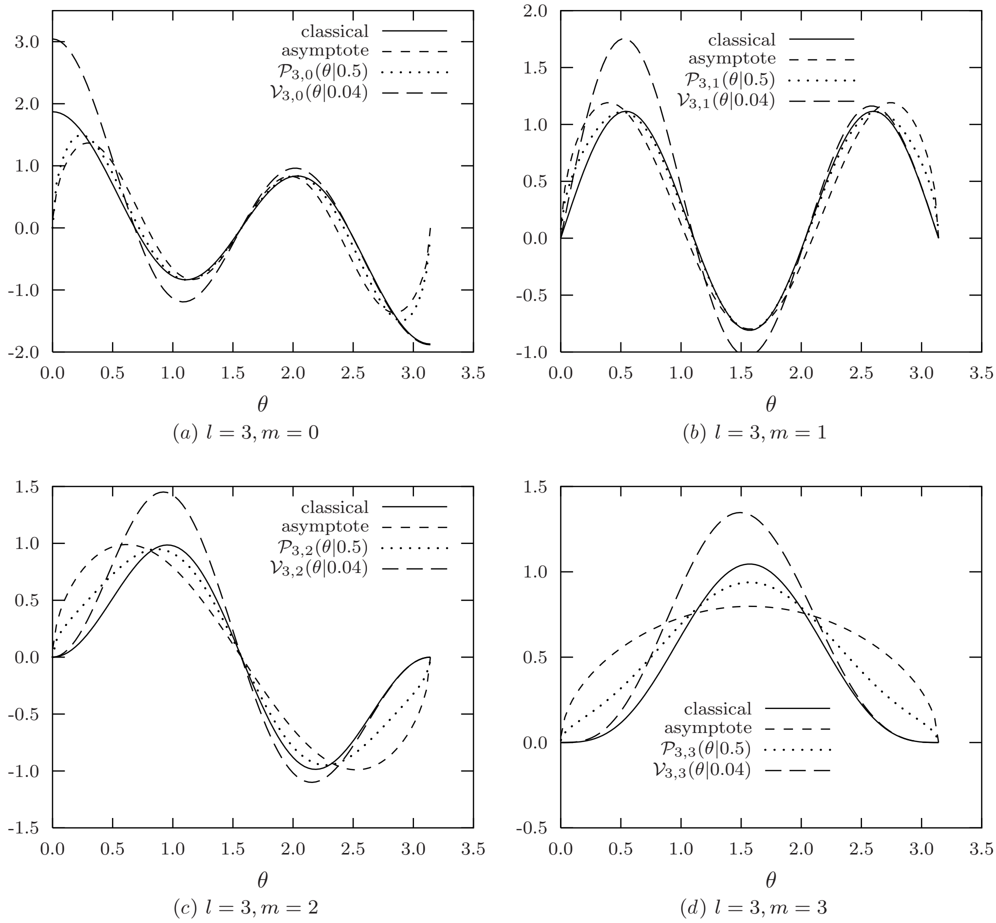


Figure 6. A comparison among the classical Legendre functions and their two q-deformed versions for $q \in \mathbb{R}$: the q-Vilenkin functions [8] $\mathcal{V}_{lm}(\theta|\omega)$ and $\mathcal{P}_{lm}(\theta|\omega)$ given in (48), where $q = \exp(\omega)$.

Table 4. Normalized q-Vilenkin functions [8] $\mathcal{V}_{lm}(x|q)$ for $l = 3, q \in \mathbb{R}$ and $x = \cos\theta$. Their normalization constant for $l = 3$ is $\mathcal{N}_3 = (4q^7 \ln q)(q^{14} - 1)^{-1}$, independently of m, and $F(x) = [(1-x)q^6 + (1+x)][(1-x)q^4 + (1+x)][(1-x)q^2 + (1+x)]$.

l	m	$\mathcal{V}_{lm}$
3	0	$2q^8[(q^8 + 2q^6 + 4q^4 + 2q^2 + 1)x^3 - (q^8 + 2q^6 + 2q^2 + 1)x]/F(x)$
3	1	$q^{15/2}\left[\prod_{j=2}^{3}\left(\sum_{k=0}^{j} q^{2k}\right)\right]^{1/2}\sqrt{1-x^2}[(q^4 + 3q^2 + 1)x^2 - (q^4 - q^2 + 1)]/F(x)$
3	2	$2q^8(q^2+1)^{-1/2}\left[\prod_{j=2}^{4}\left(\sum_{k=0}^{j} q^{2k}\right)\right]^{1/2}(1-x^2)x/F(x)$
3	3	$[(q^4+q^2+1)(q^2+1)]^{-1/2}\left[\prod_{j=2}^{5}\left(\sum_{k=0}^{j} q^{2k}\right)\right]^{1/2}(1-x^2)^{3/2}/F(x)$

while the q-Legendre functions are not defined on the unitary circle, the q-Vilenkin functions are well defined there [8]. Note also in figure 6(a) that $\mathcal{P}_{30}$ tends to its classical limit as $q \to 1$ in a non-trivial way (singular at the endpoints $\theta = 0$ and $\theta = \pi$) much like g_0 does in figure 1.

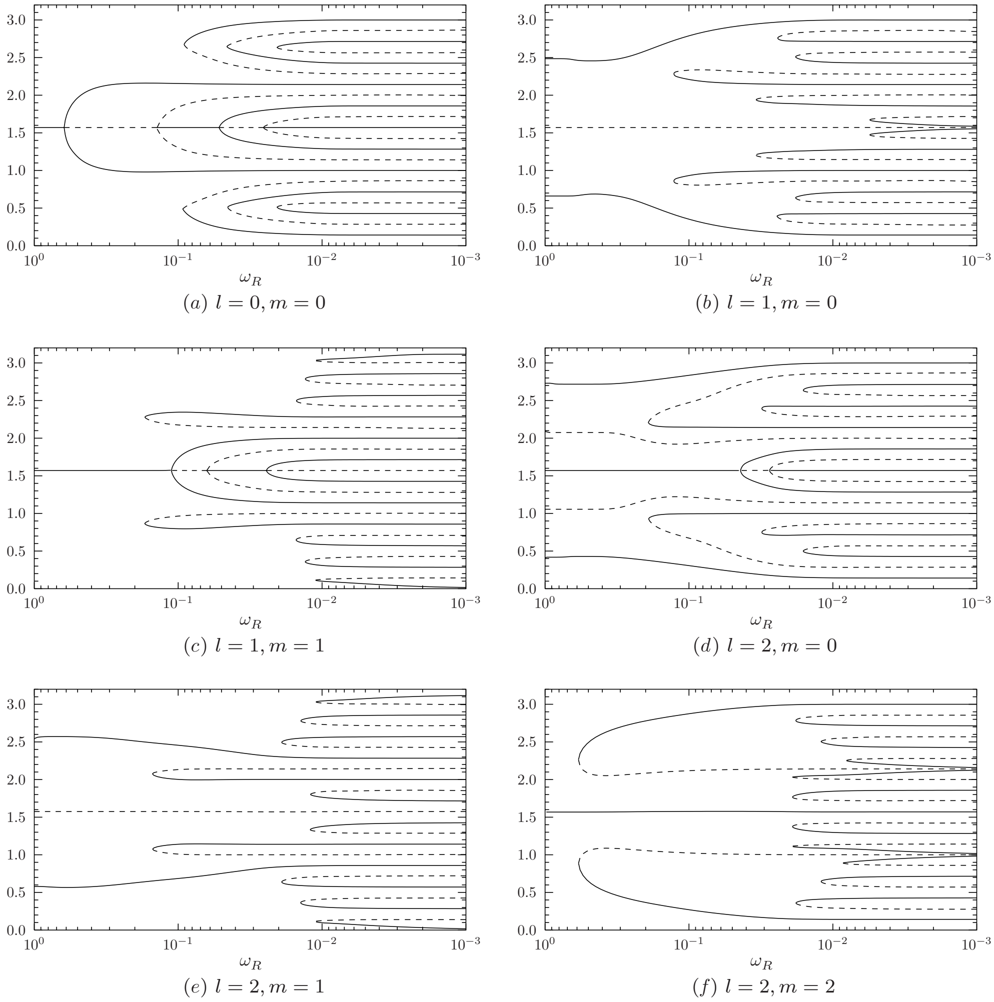


Figure 7. Bifurcation sets of $|\mathcal{P}_{lm}(\theta|\omega_R + \mathrm{i})|^2$, where θ is the vertical axis. The solid lines are the maxima and the dashed lines are the minima.

Following the lines of the discussion presented in the last section, we show in figures 7 and 8 the position of all maxima and minima of $|\mathcal{P}_{lm}(\theta|q)|^2$, $l \leqslant 3$, $q = \exp(\omega_R + \mathrm{i})$, as a function of the deformation parameter $\omega_R \to 0$. We can see in all cases that the qualitative topographic properties of these q-functions are changed as the control parameter ω_R is changed in a typical morsification process with non-Morse critical points of type A_2. In each figure, the vertical axis is the angular position θ and the solid (dashed) lines represent maxima (minima). Observe in these figures that the pitchfork bifurcations occur only in $|\mathcal{P}_{00}|^2$, $|\mathcal{P}_{11}|^2$, $|\mathcal{P}_{20}|^2$, $|\mathcal{P}_{31}|^2$ and $|\mathcal{P}_{33}|^2$, at least in region 1–0.001 shown for ω_R. We observe again the fractal-like nature of this process for smaller values of the deformation parameter ω_R and that $|\mathcal{P}_{lm}(\theta|q)|^2$ must have an infinite number of singular points when $\omega_R = 0$. The bifurcation diagram 8(b) shows a pitchfork in reverse order from all other cases. In this same diagram the minimum peaks undergo another pitchfork bifurcation in the region 0.001–0.0001, but this time in phase with all others.

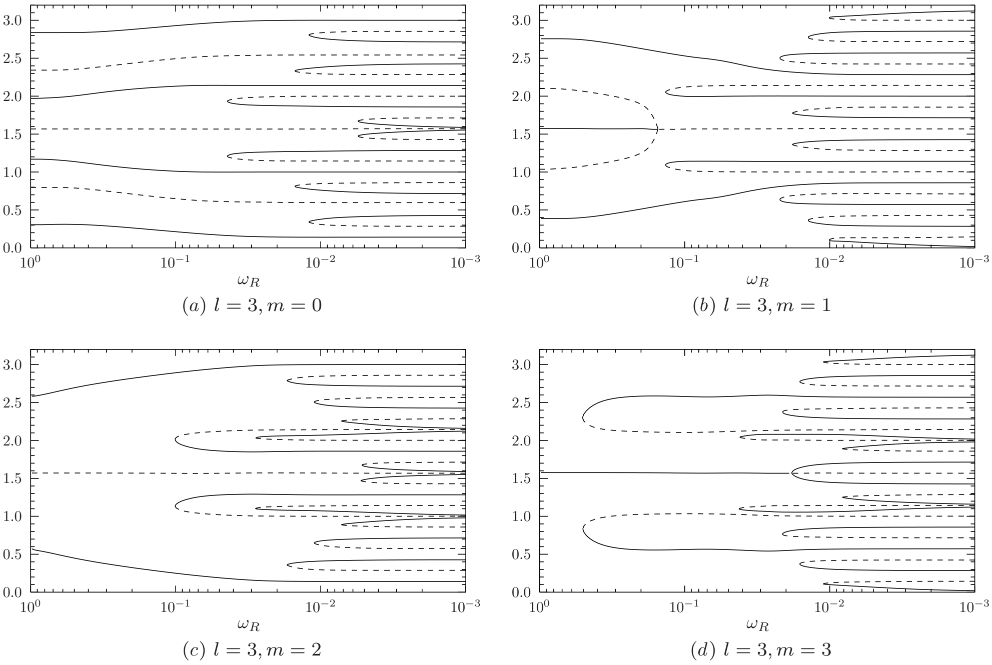


Figure 8. Continuation of the bifurcation sets of $|\mathcal{P}_{lm}(\theta|\omega_R + \mathrm{i})|^2$, where θ is the vertical axis. The solid lines are the maxima and the dashed lines are the minima.

7. Conclusions

We showed in this paper that a particular family of Askey–Wilson polynomials can be interpreted directly in the light of the q-deformed algebra $su_q(2)$. We also corrected and proved previous results concerning the q-Legendre functions such as their orthonormalization. The special cases $q \to 1$ (classical limit) and $q \to \infty$ (asymptote) were established properly.

In order to appraise the potentiality of q-deformed special functions in applications such as optimization procedures, we investigated numerically the properties of some q-Legendre functions for different values of the deformation parameter q. We conclude that the most interesting situation is when the deformation parameter approaches S^1. In that case these q-functions can exhibit a complex structure, where their singular points undergo a series of fractal-like bifurcations, qualitatively different from their classical limits.

Further investigation concerning different routes to S^1, with special attention to roots of unity, is underway for both q-Legendre and q-Vilenkin functions.

Acknowledgment

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Appendix. The first order q-differential equation for $\Phi_{l,m}$

We present here a direct verification that the basic hypergeometric (39) is the solution of the first order q-differential equation (38). Since the θ dependence on $\Phi_{l,m}(\theta)$ given in (39) comes only from the product $(q^{-(m+1)}\,\mathrm{e}^{\mathrm{i}\theta}, q^{-(m+1)}\,\mathrm{e}^{-\mathrm{i}\theta}; q^{-2})_k$, we need to evaluate this term in $\theta \pm \mathrm{i}\omega$ in order to calculate its q-derivative D_q as defined in (21):

$$D_q(q^{-(m+1)}\,\mathrm{e}^{\mathrm{i}\theta}, q^{-(m+1)}\,\mathrm{e}^{-\mathrm{i}\theta}; q^{-2})_k = -2q^{-(m+k)}[k]_q(q^{-(m+2)}\,\mathrm{e}^{\mathrm{i}\theta}, q^{-(m+2)}\,\mathrm{e}^{-\mathrm{i}\theta}; q^{-2})_{k-1}. \tag{83}$$

Therefore, the left-hand side of (38) is

$$D_q\Phi_{l,m}(\theta) = -\frac{2}{q^m}\sum_{k=1}^{l-m}\frac{(q^{2(l-m)}, q^{-2(l+m+1)}; q^{-2})_k(q^{-(m+2)}\,\mathrm{e}^{\mathrm{i}\theta}, q^{-(m+2)}\,\mathrm{e}^{-\mathrm{i}\theta}; q^{-2})_{k-1}}{(q^{-2(m+1)}, -q^{-2(m+1)}, -q^{-2(m+1)}, q^{-2}; q^{-2})_k}\frac{[k]_q}{q^{3k}}. \tag{84}$$

If we write all θ-independent Pochhammer symbols as

$$(a, b, \ldots; q)_k = (1 - q^{k-1}a)(1 - q^{k-1}b)\cdots(a, b, \ldots; q)_{k-1} \tag{85}$$

and change the sum index ($k \to k-1$), we have

$$D_q\Phi_{l,m}(\theta) = \frac{1}{2}\sum_{k=0}^{l-m}\left\{\frac{(q^{2(l-m)}, q^{-2(l+m+1)}; q^{-2})_k(q^{-(m+2)}\,\mathrm{e}^{\mathrm{i}\theta}, q^{-(m+2)}\,\mathrm{e}^{-\mathrm{i}\theta}, q^{-2})_k}{(q^{-2(m+1)}, -q^{-2(m+1)}, -q^{-2(m+1)}, q^{-2}; q^{-2})_k}q^{-k} \right. \\ \left. \times\frac{[l-m-k]_q[l+m+k+1]_q}{\cosh^2[(m+k+1)\omega][m+k+1]_q}\right\}. \tag{86}$$

As a final step, using the following properties of Pochhammer symbols

$$(s_1q^{s_2 2\zeta}; q^{-2})_k(q^{2(\zeta-s_2k)} - s_1q^{-2(\zeta-s_2k)}) = (s_1q^{s_2 2(\zeta-s_2 1)}; q^{-2})_k(q^{\zeta} - s_1q^{-\zeta}) \\ s_1 = \pm \qquad s_2 = \pm \tag{87}$$

in (86), we find the right-hand side of (38). Note the presence of $\cosh(m+k+1)\omega$ in (86), which is the motivation of our $\prod_{k=m+1}^{l}\cosh(k\omega)$ coefficient in (28). It must be remarked that this coefficient is missing in equation (4.6) of [7]. Consequently, they are not using the same $Q_{l,m}$ function in their vertical and horizontal recurrence relations.

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